3. The sheaf of regular functions

Big Picture

→ Have defined affine varieties X & their Zariski topology

→ Next: What are morphisms $X \xrightarrow{f} Y$? ⇝ category of affine varieties

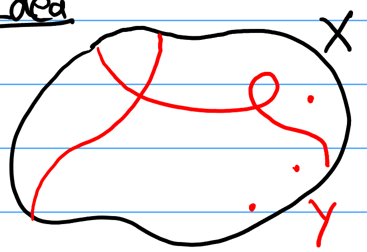
Definition should satisfy:

- f gives map of sets
- f is continuous in Zar. topology
- being a morphism can be checked locally on X (e.g. on open cover)
  ↳ Similar to: continuous, smooth, holomorphic, ...

→ Easiest case: Morphisms to $\mathbb{A}^1$ = regular functions

- morphisms $X \longrightarrow \mathbb{A}^1$ $\hat{=}$ polynomial functions $f \in A(X)$
- morphisms $U \longrightarrow \mathbb{A}^1$ for $U \subseteq X$ open ?

Idea



For $U \subseteq X$ open with complement $Y = X \setminus U$, we can allow functions
$\varphi : U \to K$, $x \mapsto \frac{g(x)}{f(x)}$ for $f, g \in A(X)$
such that $f(x) \neq 0$ for $x \in U$.

To get a definition that can be checked locally: only require that φ has this form in a neighbourhood of each point of U.

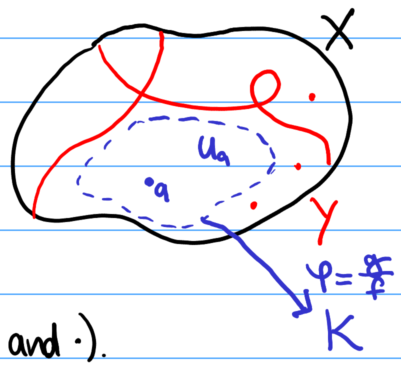
Def (Regular functions)

Let X be an affine variety and let U be an open subset of X.
A <u>regular function</u> on U is a map $\varphi : U \to K$ satisfying:

For every $a \in U$ there is an open set $a \in U_a \subseteq U$ and polynomial functions $f, g \in A(X)$ with $f(x) \neq 0$ for $x \in U_a$ and

$$\varphi(x) = \frac{g(x)}{f(x)} \quad \text{for } x \in U_a. \qquad (*)$$

The set of all such regular functions on U will be denoted $\mathcal{O}_X(U)$.

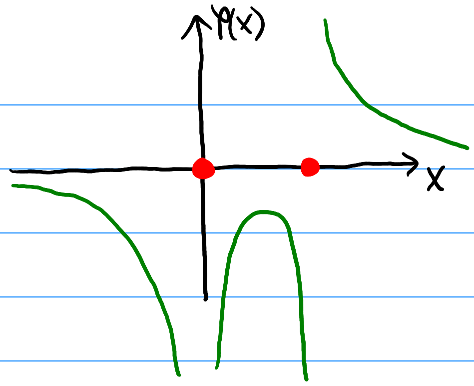


Note: $\mathcal{O}_X(U)$ is a K-algebra (with pointwise $+$ and $\cdot$).

- Write "$\varphi = g/f$ on U_a" as shorthand for $(*)$; later: connection to localization

Exa $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0,1\}$

$$\varphi : U \longrightarrow K$$
$$x \longmapsto \varphi(x) = \frac{3x^2+5}{x^2(x-1)}$$



$\rightsquigarrow$ here we can take $U_a = U$

Exa (Local $\neq$ global quotients of polynomials)

Consider $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$ and the open subset
$$U = X \setminus V(x_2, x_4) = \{(x_1,x_2,x_3,x_4) \in X : x_2 \neq 0 \text{ or } x_4 \neq 0\} \subseteq X.$$

Then
$$\varphi : U \longrightarrow K, \quad (x_1,x_2,x_3,x_4) \mapsto \begin{cases} x_1/x_2 & \text{if } x_2 \neq 0 \\ x_3/x_4 & \text{if } x_4 \neq 0 \end{cases}$$
is a regular function on U:

- Well-defined: if $x_2 \neq 0$ and $x_4 \neq 0 \Rightarrow \frac{x_1}{x_2} = \frac{x_3}{x_4}$ since $x_1x_4 = x_2x_3$.
- Locally quotient of polynomials (on $U_2 = X \setminus V(x_2)$ and $U_4 = X \setminus V(x_4)$)
- None of the quotients x_1/x_2 or x_3/x_4 makes sense on all of U (e.g. at points $(0,0,0,1)$ or $(0,1,0,0)$)
- Fact: There is no presentation g/f as quot. of polynomials which makes sense on all of U.

Aside Interpretation of φ

If we identify $\mathbb{A}^4 \cong \mathrm{Mat}_{2\times 2}(K) \Rightarrow X = V(x_1x_4 - x_2x_3)$
$$(x_1,x_2,x_3,x_4) \longmapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = M \qquad = V(\det(M))$$
$$= \{M \in \mathrm{Mat}_{2\times 2}(K) : \underline{M \text{ has rank} \leq 1}\}$$

$$\Rightarrow U = X \setminus V(x_2, x_4)$$
$$= \{M \in X : \underline{\text{second column vector } \tbinom{x_2}{x_4} \neq \tbinom{0}{0}}\}$$

first column vector $\binom{x_1}{x_3} \overset{!}{=} \lambda \cdot \binom{x_2}{x_4}$

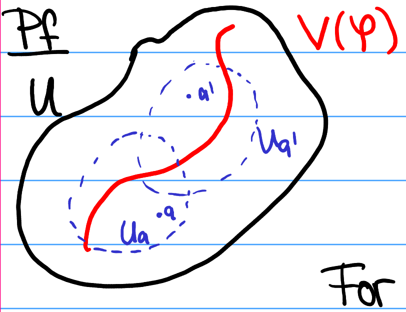
Then $\varphi\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \lambda$ is this constant!

Properties of regular functions

In this section we collect nice features of regular functions.

Lem (Zero loci of regular functions are closed)
Let U be an open subset of the affine variety X and $\varphi \in \mathcal{O}_X(U)$.
Then
$$V(\varphi) := \{x \in U : \varphi(x) = 0\}$$
is closed in U.

Pf



Fact from topology:
Can check whether $V \subseteq U$ is closed on any open cover $U = \bigcup_a U_a$ of U.

For $a \in U$ take $U_a \subseteq U$ open such that $\varphi(x) = \frac{g(x)}{f(x)}$

$$\Rightarrow V(\varphi) \cap U_a = \{x \in U_a : \tfrac{g(x)}{f(x)} = 0\} = \{x \in U_a : g(x) = 0\}$$
$$= \underbrace{V(g)}_{\text{closed}} \cap U_a \quad \text{is closed in } U_a$$

Furthermore: $\bigcup_{a \in U} U_a = U$ since $a \in U_a$. $\square$

Given $U \subseteq V \subseteq X$ two open subsets of an affine var. X, there is a well-defined <u>restriction map</u>

$$\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U), \quad (\varphi : V \to K) \longmapsto (\varphi|_U : U \to K).$$

check: still locally given by g/f.

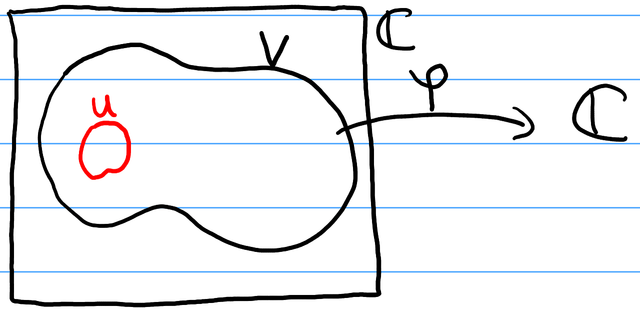
In general this is not surjective
(e.g. for $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) \longrightarrow \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\})$: $\varphi(x) = \frac{1}{x}$ in $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\})$ is not in image)
But the following result shows that it is injective with right assumpt!

Pro (Identity theorem for regular functions)
Let X be irreducible and $\emptyset \neq U \subseteq V \subseteq X$ open subsets.
For $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ with $\varphi_1|_U = \varphi_2|_U$ one has $\varphi_1 = \varphi_2$.

Pf $V(\underbrace{\varphi_1 - \varphi_2}_{\in \mathcal{O}_X(V)}) \subseteq V$ closed and $U \subseteq V(\varphi_1 - \varphi_2) \xRightarrow[\text{dense}]{U \subseteq X} \overline{U} = V \subseteq V(\varphi_1 - \varphi_2)$

$\Rightarrow \varphi_1 = \varphi_2$ on all of V. $\square$

Rmk In complex analysis   **Identity theorem**



$\varphi: V \to \mathbb{C}$ holomorphic (connected)

$\emptyset \neq U \subseteq V$ open with

$\varphi|_U = 0$, then $\varphi = 0$

φ uniquely determined by values on U

The above is an analogue in alg. geometry.

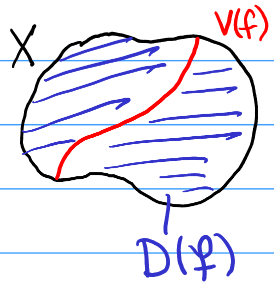
Distinguished open subsets

Q How to compute $\mathcal{O}_X(U)$ in practice?

We start with the most basic open sets $U \subseteq X$:

Def (Distinguished open subsets)

For an affine variety X and a polynomial function $f \in A(X)$, we call



$$D(f) := X \setminus V_X(f) = \{x \in X : f(x) \neq 0\}$$

the distinguished open subset of f in X.

Rmk (Properties of distinguished subsets)

- Finite intersections: $D(f) \cap D(g) = \{x \in X : f(x) \neq 0, g(x) \neq 0\}$
$$= \{x \in X : (f \cdot g)(x) \neq 0\} = D(f \cdot g)$$

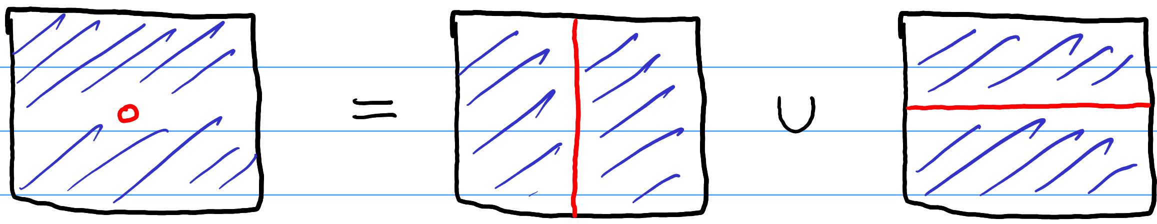
- Finite unions:

Any open subset $U \subseteq X$ is a finite union of distinguished subsets:

$$U = X \setminus \underbrace{V_X(f_1, \ldots, f_m)}_{\text{any closed subset is zero locus of fin. many functions}} = X \setminus (V_X(f_1) \cap \cdots \cap V_X(f_m))$$
$$= (X \setminus V_X(f_1)) \cup (X \setminus V_X(f_2)) \cup \cdots \cup (X \setminus V_X(f_m))$$
$$= D(f_1) \cup D(f_2) \cup \cdots \cup D(f_m)$$

$\Rightarrow \{D(f) : f \in A(X)\}$ is a basis of the Zariski top.

Exa



$$U = \mathbb{A}^2 \setminus \{(0,0)\} = D(x) \cup D(y)$$

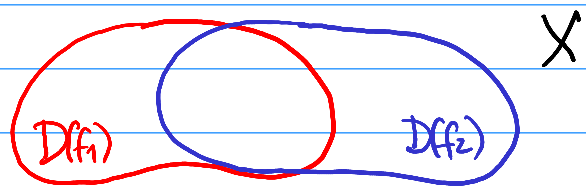
Big picture To understand $\mathcal{O}_X(U)$
- First: Compute $\mathcal{O}_X(D(x))$, $\mathcal{O}_X(D(y))$
- Then: $f: U \to K$ is regular if $f|_{D(x)} \in \mathcal{O}_X(D(x))$ and $f|_{D(y)} \in \mathcal{O}_X(D(y))$

To compute $\mathcal{O}_X(D(f))$, we'll use the following important tool:

Lem (Generalized partition of unity)
Inside the affine variety X, assume
$D(f) = \bigcup_{a \in B} D(f_a)$ $(\star)$ for some $f_a \in A(X)$, $a \in B$. some index set
Then there exists $m \in \mathbb{N}$ such that f^m is a finite sum

$$f^m = \sum_{a \in B_0} r_a \cdot f_a \quad \text{for } B_0 \subseteq B \text{ finite, } r_a \in A(X).$$

↳ taking $f = 1$:
$X = \bigcup_a D(f_a)$



$\Rightarrow 1 = \sum r_a \cdot f_a$.

$$1 = \underbrace{r_1 \cdot f_1}_{\text{supp. on } D(f_1)} + \underbrace{r_2 \cdot f_2}_{\text{supp. on } D(f_2)}.$$

Pf Taking the complement of both sides of $(\star)$:

$$V_X(f) = \bigcap_{a \in B} V_X(f_a) = V_X(\langle f_a : a \in B \rangle).$$

$$\xrightarrow[\text{Satz}]{\text{Nullstellen-}} f \in I_X(V_X(f)) = I_X(V_X(\langle f_a : a \in B \rangle)) = \sqrt{\langle f_a : a \in B \rangle}$$

$$\Rightarrow \exists\, m \in \mathbb{N}: \ f^m \in \langle f_a : a \in B \rangle = \Big\{ \sum_{\text{finite}} r_a f_a \Big\}. \qquad \square$$

Regular functions on distinguished open subsets

Pro (Regular functions on distinguished open sets)
Let X be an affine variety and $f \in A(X)$. Then

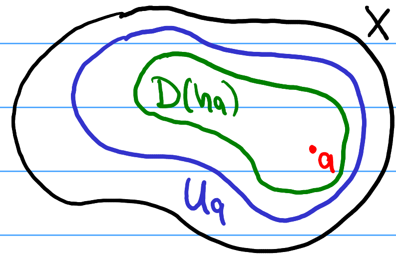
$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X),\ m \in \mathbb{N} \right\}$$

In particular:

- for $f = 1$, we have $D(f) = X$ and $\mathcal{O}_X(X) = \{g : g \in A(X)\} = A(X)$, so the regular functions on all of X are exactly the polynomial functions.
- on a distinguished open set, a regular function is always globally the quotient of two polynomial functions

Proof "$\supseteq$" $(f^m)(x) \neq 0$ for $x \in D(f)$, so g/f^m is well-defined and a regular function on $D(f)$ by definition.

"$\subseteq$" Let $\varphi : D(f) \to K$ be a regular function.
By definition: for all $a \in D(f)$ we can write $\varphi = \frac{g_a}{f_a}$ for some $f_a, g_a \in A(X)$ on some open neighborhood $a \in U_a \subseteq X$.
$f_a(x) \neq 0$ for $x \in U_a$



↳ Since dist. opens form a basis of the Zar. topology: can shrink neighborhood and assume $U_a = D(h_a)$ for $h_a \in A(X)$.

↳ replacing $\varphi = \frac{g_a}{f_a} = \frac{g_a \cdot h_a}{f_a \cdot h_a}$ $\rightsquigarrow$ can assume that numerator g_a and denominator f_a of φ vanish on $X \setminus D(h_a) = V_X(h_a)$.

This means: $f_a(x) \begin{cases} \neq 0, & x \in D(h_a) \\ = 0, & x \in X \setminus D(h_a) \end{cases} \Rightarrow D(h_a) = D(f_a)$

↳ replace $h_a = f_a$ $\rightsquigarrow$ this does not change $D(h_a)$.

Summary For $a \in D(f)$ have $\varphi = \frac{g_a}{f_a}$ on $D(f_a) \subseteq D(f)$ with $g_a|_{X \setminus D(f_a)} = 0$.

Claim For $a, b \in D(f)$ we have: $\boxed{g_a \cdot f_b = g_b \cdot f_a \in A(X) \quad (*)}$

Indeed:

→ for $x \in D(f_a) \cap D(f_b) = X \setminus (V_X(f_a) \cup V_X(f_b))$ we have
$g_a(x)/f_a(x) = \varphi(x) = g_b(x)/f_b(x) \Rightarrow g_a(x) f_b(x) = g_b(x) f_a(x)$.

→ for $x \in V_X(f_a)$: $g_a(x) = f_a(x) = 0 \Rightarrow$ both sides of $(*)$ vanish at x

→ similar for $x \in V_X(f_b)$. ⁂

To finish the proof:

$$D(f) = \bigcup_{a \in D(f)} D(f_a) \xRightarrow[\text{of unity)}]{\text{Lem (Part.}} f^m = \sum_{a \,\leftarrow \text{finite}} r_a \cdot f_a \;, r_a \in A(X)$$

Set $g := \sum_a r_a \cdot g_a$, then we claim: $\varphi = g/f^m$ on $D(f)$.

Indeed: for $b \in D(f)$ we have $\varphi = \frac{g_b}{f_b}$ and

$$g \cdot f_b = \sum r_a \cdot g_a \cdot f_b \overset{(*)}{=} \sum r_a \cdot g_b \cdot f_a = g_b \cdot f^m \quad \text{on } D(f_b)$$

$$\Rightarrow \varphi = \frac{g_b}{f_b} = \frac{g}{f^m}. \qquad \square$$

Rmk

In the proof we used again that K is alg. closed.
(via Lemma (partit. of unity)).
In fact the statement of the Prop. (e.g. $\mathcal{O}_X(X) = A(X)$) is false otherwise:
For $K = \mathbb{R}$, $X = \mathbb{A}^1_{\mathbb{R}}$ we have $f = \frac{1}{x^2+1}$ is a regular function on X, but not globally a polynomial.

Regular functions via localization

Saw that on $D(f)=\{x\in X : f(x)\neq 0\}$ every regular function has the form

$$\varphi = \frac{g}{f^m} \quad \text{for } g\in A(X),\ m\in\mathbb{N}.$$

Such quotients appear naturally in commutative algebra.

Localization
R ring, $S\subseteq R$ multiplicative (i.e. $s_1\cdot s_2\in S$ for $s_1,s_2\in S$)
$\rightsquigarrow$ $S^{-1}R = R[S^{-1}] = \{r/s : r\in R, s\in S\}/\sim$ localization of R at S
where
$r_1/s_1 \sim r_2/s_2$ if $\exists\, t\in S: \ t\cdot(r_1 s_2 - r_2 s_1) = 0$.
For $f\in R$ and $S=\{1,f,f^2,f^3,\dots\}\subseteq R$ denote $R_f := R[S^{-1}]$.

Cor (Regular functions as localizations)
Let X be an affine variety and let $f\in A(X)$.
Then
$$\mathcal{O}_X(D(f)) \cong A(X)_f \quad \text{as } K\text{-algebras.}$$

Pf Consider the K-algebra morphism

$$A(X)_f \xrightarrow{\ \Phi\ } \mathcal{O}_X(D(f)), \quad \underbrace{\frac{g}{f^m}}_{\text{formal symbol}} \longmapsto \underbrace{\frac{g}{f^m}}_{\text{function } D(f)\to K,\ x\mapsto \frac{g(x)}{f(x)^m}}$$

Well-defined: $\frac{g_1}{f^r}=\frac{g_2}{f^s}$ in $A(X)_f \iff \exists\, m\geq 0: \quad f^m\cdot(g_1 f^s - g_2 f^r)=0$
Then for each $x\in D(f)$:

$$\underbrace{f(x)^m}_{\neq 0}\cdot(g_1(x) f(x)^s - g_2(x) f(x)^r) = 0 \Rightarrow g_1(x) f(x)^s = g_2(x) f(x)^r$$
$$\Rightarrow g_1(x)/f(x)^r = g_2(x)/f(x)^s.$$

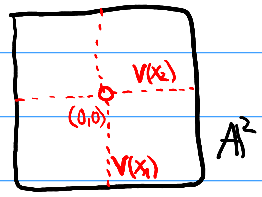
Φ is surjective by previous proposition.
Φ is injective: Assume $\Phi(g/f^m)=0 \rightsquigarrow g(x)/f(x)^m = 0 \ \forall x\in D(f)$
$\rightsquigarrow g(x)=0 \ \forall x\in D(f) \rightsquigarrow (f\cdot g)(x)=0 \ \forall x\in X \rightsquigarrow f\cdot g = 0\in A(X)$
$\Rightarrow \frac{g}{f^m}=\frac{0}{1}$ in $A(X)_f$ since $f\cdot(g\cdot 1 - f^m\cdot 0) = 0\in A(X)$. $\square$

Extending regular functions over sets of high codimension

After understanding $\mathcal{O}_X(U)$ for $U = D(f)$, let's see an example of a set U not of this form:

Exa (Regular functions on $U = \mathbb{A}^2 \setminus \{(0,0)\} \subseteq \mathbb{A}^2$)

We claim that for $U = \mathbb{A}^2 \setminus \{(0,0)\}$:



$$\mathcal{O}_{\mathbb{A}^2}(U) = K[x_1, x_2] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2).$$

In other words: $\boxed{\text{Every regular function on } \mathbb{A}^2\setminus\{(0,0)\} \text{ extends over } (0,0).}$

Similar to Removable Singularity thm / Hartog's thm: "Every holom. fct. on $\mathbb{C}^2 \setminus \{(0,0)\}$ can be extended holomorph. to $\mathbb{C}^2$."

Indeed, let $\varphi \in \mathcal{O}_{\mathbb{A}^2}(U)$. From $U = D(x_1) \cup D(x_2)$ we have:

$$\varphi|_{D(x_i)} \in \mathcal{O}_{\mathbb{A}^2}(D(x_i)) \overset{\text{Pro}}{\Rightarrow} \begin{cases} \varphi = f/x_1^m & \text{on } D(x_1) \\ \varphi = g/x_2^n & \text{on } D(x_2) \end{cases} \quad \text{for } f, g \in K[x,y]$$

Wlog: assume $x_1 \nmid f$ and $x_2 \nmid g$.

$\Rightarrow f/x_1^m = g/x_2^n$ in $\mathcal{O}_{\mathbb{A}^2}(D(x_1) \cap D(x_2)) = \mathcal{O}_{\mathbb{A}^2}(D(x_1x_2)) = K[x_1,x_2]_{x_1x_2}$

$$\Rightarrow \exists r \in \mathbb{N}: \quad (x_1x_2)^r \cdot (f \cdot x_2^n - g \cdot x_1^m) = 0 \in K[x_1,x_2]$$

$$\xrightarrow[\text{domain}]{K[x_1,x_2]} \quad f \cdot x_2^n = g \cdot x_1^m \quad \in K[x_1,x_2]$$

$x_1 \nmid f$ and $x_1 \nmid x_2$ so for $m > 0$ we would get a contradict. ($K[x_1,x_2]$ is UFD)

$\Rightarrow m = n = 0$ and $\varphi = f = g \in K[x_1,x_2]$. $\square$

Exercise Let Y be a non-empty irred. subvar. of an affine var. X. Set $U = X \setminus Y$. [above: $Y = \{(0,0)\} \subseteq X = \mathbb{A}^2$].

(a) Assume that $A(X)$ is a UFD. Show that $\mathcal{O}_X(U) = A(X)$ if and only if $\operatorname{codim} Y \geq 2$.

Slogan On nice spaces X, regular fcts. extend over sets of codim ≥ 2.

(b) Show by example that (a) is false without UFD assumption.

Sheaves

Have seen that regular functions on $V \subseteq X$

↳ restrict to regular functions on open subsets $U \subseteq V$,

↳ $\varphi: V \to K$ being regular is local condition on V,

↳ can be specified on an open cover $V = \bigcup_i V_i$
by a family $\varphi_i \in \mathcal{O}_X(V_i)$ such that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$.

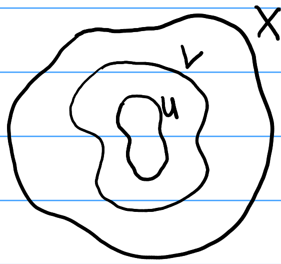
Such phenomena occur in many parts of mathematics
⇝ develop a formalism for rest of chapter.

<u>Def</u> (Presheaves) Given a top. space X, a <u>presheaf</u> $\mathcal{F}$ of rings on X consists of the data:

- for every open set $U \subseteq X$ a ring $\mathcal{F}(U)$ think: functions on U
- for every inclusion $U \subseteq V$ of open sets in X a ring homomorphism

$$\rho_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$

think: restriction of functions from V to U.



called the <u>restriction map</u>.

Such that:

- $\rho_{U,U}$ is the identity map on $\mathcal{F}(U)$,
- for a chain $U \subseteq V \subseteq W$ of open inclusions, we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\rho_{W,V}"] \arrow[rr, bend right, "\rho_{W,U}"'] & \mathcal{F}(V) \arrow[r, "\rho_{V,U}"] & \mathcal{F}(U) \end{tikzcd}$$

The elements $\varphi \in \mathcal{F}(V)$ are usually called the <u>sections</u> of $\mathcal{F}$ on V, and $\rho_{V,U}$ is written as $\rho_{V,U}(\varphi) = \varphi|_U$.

<u>Note</u> [Gathmann] requires $\mathcal{F}(\emptyset) = 0$ in addition, but I don't think that's necessary (or standard).

Exa Let X be any topological space

(a) The presheaf $\mathcal{F} = C(-, \mathbb{R})$ of continuous functions to $\mathbb{R}$ is given by

$$\mathcal{F}(U) = \{ \varphi : U \to \mathbb{R} : \varphi \text{ continuous} \}$$

with the usual restriction map $\varphi \mapsto \varphi|_{U}$.

[for $X = \mathbb{R}^n$: can also take "analytic", "differentiable", "arbitrary" fcts here]

(b) Given any ring R, the presheaf $\mathcal{G}$ of constant funct. to R is given by

$$\mathcal{G}(U) = \{ \varphi : U \to R : \varphi \text{ is constant} \}$$

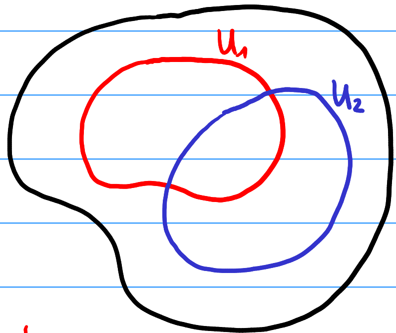
with usual restriction.

i.e. $\exists a \in R : \varphi(x) = a \ \forall x \in U$

Def (Sheaves) A presheaf $\mathcal{F}$ is called a sheaf of rings if it satisfies the following gluing property:

If

- $U \subseteq X$ is an open set
- $\{U_i : i \in I\}$ an arbitrary open cover of U
- $\varphi_i \in \mathcal{F}(U_i)$ sections for all $i \in I$



such that

$$\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j} \quad \forall i, j \in I$$

$\hookrightarrow$ φ on $U_1 \cup U_2$
$\Leftrightarrow$ φ_1 on U_1, φ_2 on U_2 w/ $\varphi_1 = \varphi_2$ on $U_1 \cap U_2$.

then there is a unique $\varphi \in \mathcal{F}(U)$ with $\varphi|_{U_i} = \varphi_i \ \forall i \in I$.

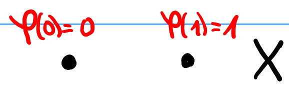
existence   uniqueness

Exa (a) $\mathcal{F} = C(-, \mathbb{R})$ is a sheaf (cont. fcts. glue on open cover)

(b) $\mathcal{G}$ given by constant funct. to R is not in general a sheaf!

Exa $X = \{0, 1\} \subseteq \mathbb{R}$, $R = \mathbb{Z}$

$\rightsquigarrow$ $U = X = \underbrace{\{0\}}_{U_0} \cup \underbrace{\{1\}}_{U_1}$ is open cover



$$\varphi_0 = 0 \in \mathcal{G}(U_0), \ \varphi_1 = 1 \in \mathcal{G}(U_1) \rightsquigarrow \varphi_0|_{U_0 \cap U_1} = \varphi_0|_{\emptyset} = \varphi_1|_{\emptyset}.$$

But there is no const. fct. $\varphi : U \to \mathbb{Z}$ restrict. to φ_0, φ_1.

$\rightsquigarrow$ being constant cannot be checked locally on X!

<u>RmK</u> (Sheaves over other categories)

Above: defined (pre-)sheaves of rings ↗ See Appendix for reminder

More generally, given a suitable category $\mathcal{C}$ we have (pre-)sheaves with values in $\mathcal{C}$

(e.g. $\mathcal{C}$ = K-algebras, Abelian groups, K-Modules, sets, ...)

↳ $\mathcal{F}(U)$ are objects of $\mathcal{C}$

↳ $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ are morphisms in $\mathcal{C}$

<u>Exa</u> Let X be an affine variety. Then the rings $\mathcal{O}_X(U)$ for $U \subseteq X$ open with the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for $U \subseteq V$ form a sheaf of K-algebras on X.

⌈ restrict. of functions always satisfies presheaf axioms, regular function on $U = \bigcup U_i$ uniquely det. by $\varphi_i \in \mathcal{O}_X(U_i)$ which agree on overlaps

$\leadsto \varphi(x) = \varphi_i(x)$ for $x \in U_i$ is well-def. and regular ⌋

<u>Def</u> (Restrictions of (pre-)sheaves)

$\mathcal{F}$ presheaf on X and $W \subseteq X$ open. Then the <u>restriction</u> of $\mathcal{F}$ to W is defined as the presheaf $\mathcal{F}|_W$ on W with

$$\mathcal{F}|_W(U) = \mathcal{F}(U) \quad \text{for } U \subseteq W \text{ open}$$

and the restriction maps taken from $\mathcal{F}$.

<u>Bookkeeping exercise</u>

Show that if $\mathcal{F}$ is a sheaf, then so is $\mathcal{F}|_W$.

Stalks of presheaves

Intuition
- ↳ sections of presheaf $\mathcal{F}$ on $U \subseteq X$ are like functions on U
- ↳ given $a \in X$, the stalk of $\mathcal{F}$ at a describes possible functions on (small) neighborhoods of a

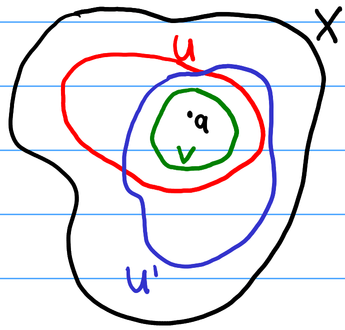
Sheaf = collection of stalks

Construction (Stalks of (pre-)sheaves)
Let $\mathcal{F}$ be a presheaf on a topological space X, and $a \in X$.
Then the <u>stalk</u> of $\mathcal{F}$ at a is defined as

$$\mathcal{F}_a = \{(U, \varphi) : U \subseteq X \text{ open with } a \in U,\ \varphi \in \mathcal{F}(U)\} / \sim$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists $V \subseteq X$ open w/ $a \in V \subseteq U \cap U'$, such that

$$\varphi|_V = \varphi'|_V .$$



Check $\sim$ is an equivalence relation and $\mathcal{F}_a$ is a K-algebra

$$[(U_1, \varphi_1)] + [(U_2, \varphi_2)] = [(U_1 \cap U_2, \varphi_1 + \varphi_2)]$$

Elements of $\mathcal{F}_a$ are called <u>germs</u> of $\mathcal{F}$ at a.
For $\mathcal{F} = \mathcal{O}_X$, X affine var $\rightsquigarrow$ write $\mathcal{O}_{X,a}$ for stalk of $\mathcal{O}_X$ at a

Rmk Germs of sheaf $\mathcal{F}$ at a
$\widehat{=}$ sections of $\mathcal{F}$ on arbitrarily small open nbhds. of a
$\rightsquigarrow$ sometimes call these local sections of $\mathcal{F}$ at a

(a) $X = \mathbb{R}$ with sheaf $\mathcal{F}(U) = \{\varphi : U \to \mathbb{R} : \varphi \text{ differentiable}\}$
$\rightsquigarrow$ have well-defined maps

$$ev_a : \mathcal{F}_a \longrightarrow \mathbb{R}, \; [(U, \varphi)] \longmapsto \varphi(a)$$
$$\partial_a : \mathcal{F}_a \longrightarrow \mathbb{R}, \; [(U, \varphi)] \longmapsto \varphi'(a)$$

but for $b \neq a$ there is no well-def. map $ev_b : \mathcal{F}_a \to \mathbb{R}$.

On the other hand: $X = \mathbb{C}$ with sheaf

$$\mathcal{G}(U) = \{\varphi : U \longrightarrow \mathbb{C} : \varphi \text{ holomorphic}\}.$$

Then any section $\varphi \in \mathcal{G}(B_1(0))$ is uniquely $\rightsquigarrow$ identity theorem.
determined by its germ $[(B_1(0), \varphi)] \in \mathcal{G}_0$ at $a = 0$.

(b) For X affine variety, $a \in X$ we do have an evaluation map

$$ev_a : \mathcal{O}_{X,a} \longrightarrow K, \; [(U, \varphi)] \longmapsto \varphi(a).$$

Stalks of affine varieties

Have defined: stalk $\mathcal{F}_x$ of sheaf $\mathcal{F}$ at point $x \in X$

Q What is the stalk $\mathcal{O}_{X,a}$ for X aff. variety?

$\rightsquigarrow$ Intuitively: should be regular functions g/f that make sense in a neighborhood of $a \in X$, i.e. satisfying $f(a) \neq 0$.

Localization at prime ideals

R ring, $P \subseteq R$ prime ideal $\Rightarrow S_P = R \setminus P \subseteq R$ is multiplicative

$(a, b \notin P \Rightarrow a \cdot b \notin P)$

Denote $R_P := R[S_P^{-1}]$ the localization at S_P

Lem For $\pi: R \to R_P$, the maps

$$\{\text{prime ideals } q \subseteq R_P\} \underset{q' \cdot R_P \leftarrow\!\shortmid\, q'}{\overset{q \mapsto \pi^{-1}(q)}{\rightleftarrows}} \{\text{prime ideals } q' \subseteq R \text{ contained in } P\}$$

are inclusion-preserving bijections inverse to each other.

$\Rightarrow R_P$ is a local ring i.e. has unique max. ideal given by $P \cdot R_P$.

Lem/Def (Stalks of regular functions as localizations)

Let $a \in X$ be a point in an affine variety. Then the map

$$A(X)_{I_X(a)} \longrightarrow \mathcal{O}_{X,a}, \quad \frac{g}{f} \mapsto \left[\left(D(f), \frac{g}{f}\right)\right]$$

$f \notin I_X(a) \Leftrightarrow f(a) \neq 0$

is an isomorphism. In particular

$$\mathcal{O}_{X,a} = \left\{ \frac{g}{f} : f, g \in A(X) \text{ with } f(a) \neq 0 \right\}.$$

As a consequence, $\mathcal{O}_{X,a}$ is a local ring with unique maximal ideal

$$I_a = \{[(U, \varphi) \in \mathcal{O}_{X,a} : \varphi(a) = 0\} \cong \left\{ \frac{g}{f} : f, g \in A(X) \text{ w/ } \begin{array}{l} g(a) = 0 \\ f(a) \neq 0 \end{array} \right\}.$$

It is called the local ring of X at a.

Proof: First we check that the map

$$A(X)_{I_X(a)} \xrightarrow{\Psi} \mathcal{O}_{X,a}\,, \quad \frac{g}{f} \mapsto \left[\left(D(f), \frac{g}{f}\right)\right]$$

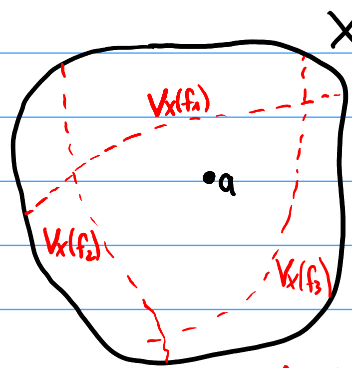
is a well-defined K-algebra morphism.

↳ $f \notin I_X(a) \Rightarrow f(a) \neq 0 \Rightarrow a \in D(f)$ and $\frac{g}{f} \in \mathcal{O}_X(D(f))$

↳ Assume that

$g_1/f_1 = g_2/f_2 \in A(X)_{I_X(a)} \Rightarrow \exists f_3 \in A(X)\setminus I_X(a)$ s.t.

$$f_3\cdot(g_1 f_2 - g_2 f_1) = 0 \in A(X)$$



For $x \in U = D(f_1\cdot f_2\cdot f_3)$:

$$\underbrace{f_3(x)}_{\neq 0}\cdot\left(g_1(x)\cdot\underbrace{f_2(x)}_{\neq 0} - g_2(x)\underbrace{f_1(x)}_{\neq 0}\right) = 0 \in K$$

$$\xrightarrow[f_3(x)f_2(x)f_1(x)]{\text{divide by}} \quad \frac{g_1(x)}{f_1(x)} = \frac{g_2(x)}{f_2(x)} \Rightarrow \left[\left(D(f_1), \frac{g_1}{f_1}\right)\right] = \left[\left(U, \frac{g_1}{f_1}\right)\right]$$

equival. relat. on $\mathcal{O}_{X,a}$ allows restriction to smaller nbhd. of a

$$= \left[\left(U, \frac{g_2}{f_2}\right)\right] = \left[\left(D(f_2), \frac{g_2}{f_2}\right)\right]$$

$\Rightarrow \Psi$ is well-defined

Ψ surjective: $[(U,\varphi)] \in \mathcal{O}_{X,a}$

⇝ Since sets $D(f)$ form basis of topology: $\exists f \in A(X): a \in D(f) \subseteq U$

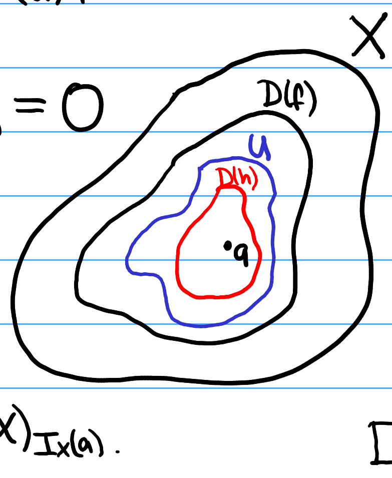
$\Rightarrow [(U,\varphi)] = [(D(f), \varphi|_{D(f)})]$ with $\varphi|_{D(f)} \in \mathcal{O}_X(D(f)) \overset{\text{Pro}}{=} A(X)_f$

$\Rightarrow \varphi = g/f^m$ is in image of Ψ since $(f^m)(a) \neq 0$.

Ψ injective: Assume $\Psi\left(\frac{g}{f}\right) = \left[\left(D(f), \frac{g}{f}\right)\right] = 0$

$\Rightarrow \exists\, a \in U \subseteq X: \quad \frac{g}{f}\big|_U = 0$

$\xRightarrow[U]{\text{shrink}} \exists h \in A(X)\setminus I_X(a): \frac{g}{f}\big|_{D(h)} = 0 \in \mathcal{O}_X(D(h)) = A(X)_h$



$\Rightarrow \exists m \in \mathbb{N}: \underbrace{h^m}_{\in A(X)\setminus I_X(a)}\cdot g = 0 \in A(X)$

$$\Rightarrow \frac{g}{f} = 0 \in A(X)_{I_X(a)}. \qquad \square$$

Appendix Categories

Many areas of mathematics are centered around some classes of objects & morphisms between them:

→ Linear algebra : Vector spaces V & linear maps $V \to W$
→ Topology : Topol. spaces X & contin. maps $X \to Y$
→ Group theory : Groups G & group hom.s $G \to H$
→ Comm. algebra : Rings R & ring hom.s $R \to S$
R-modules M & module hom.s $M \to N$

The notion of a category captures some basic familiarities between these examples:

Def A category $\mathcal{C}$ is the data of

"locally small category"

- a collection (or class) of objects $A \in \mathcal{C}$
- a set $\mathrm{Mor}(A,B)$ of morphisms for each pair $A,B \in \mathcal{C}$
- a composition map
  $$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \xrightarrow{(\varphi,\psi)\mapsto \psi\circ\varphi} \mathrm{Mor}(A,C) \quad \text{for any } A,B,C \in \mathcal{C}$$
- a distinguished identity morphism $id_A \in \mathrm{Mor}(A,A)$ for $A \in \mathcal{C}$

satisfying

- $(\psi\circ\varphi)\circ\xi = \psi\circ(\varphi\circ\xi)$ (associativity)
- $\varphi\circ id_A = \varphi$, $id_B\circ\varphi = \varphi$ for $\varphi \in \mathrm{Mor}(A,B)$ (left & right ident.)

Advantages of categories

→ automatically define notions like isomorphisms:
$\varphi \in \mathrm{Mor}(A,B)$ is isomorph. $\Leftrightarrow \exists \psi \in \mathrm{Mor}(B,A)$: $\psi\circ\varphi = id_A$, $\varphi\circ\psi = id_B$

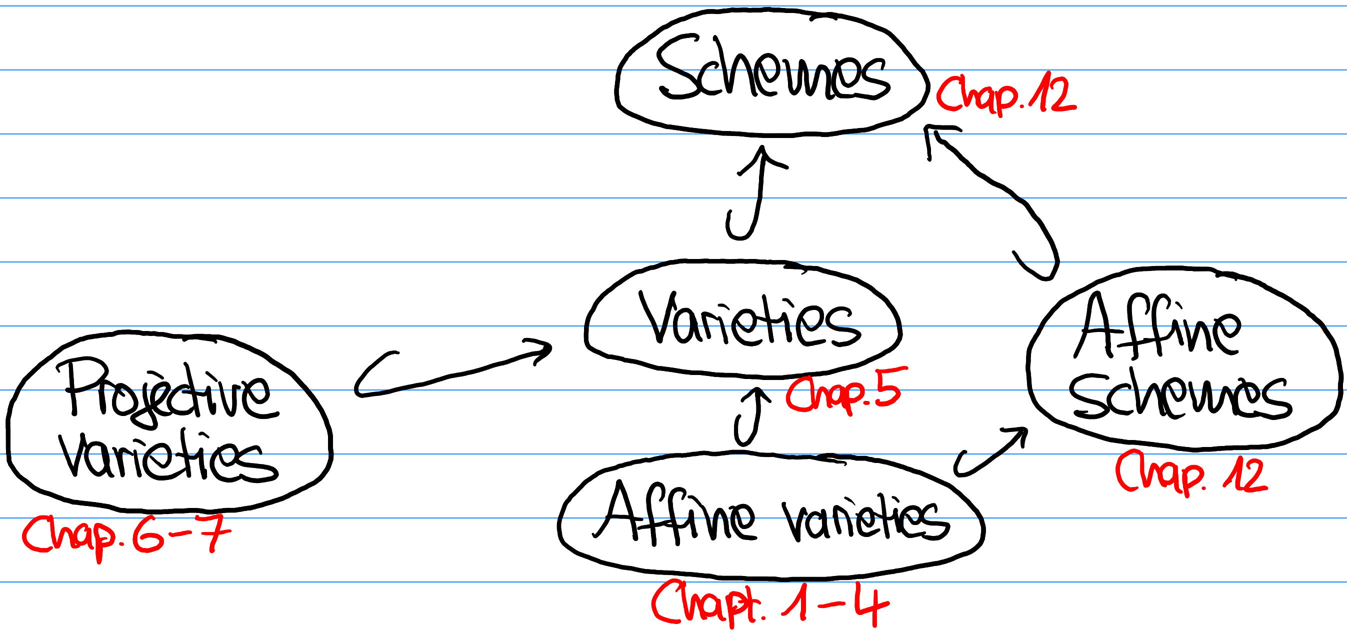
→ make connections between different areas of mathematics via functors

e.g.
$$\text{Topological Spaces} \longrightarrow \text{Groups}, \quad X \longmapsto H_*(X)$$

homology group

View of the course

Large parts of the material are about defining categories of nice geometric objects



Goal of next chapter

Define morphisms $\mathrm{Mor}(X,Y)$ between affine varieties X,Y.